Computers and Mathematics with Applications 58 (2009) 2081–2083

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Editorial

New analytical methods for cleaning up the solution of nonlinear equations

J.-H. He [a,b,*,1], E.W.M. Lee [b]

[a] Donghua University, 1882 Yan'an Xilu Road, Shanghai 200051, China

[b] City University of Hong Kong, Department of Building & Construction, Hong Kong, China

A B S T R A C T

Analytical methods belong to perhaps the most challenging, promising, and 'romantic' area of modern mathematics, and are playing an even more important role in the mathematics and other ramifications of science, prompting a resurgence of interest in the application of modern as well as classical or ancient mathematics to the search for approximate analytical solutions for various real-life nonlinear physical problems. MatLab and other mathematical software give rise to the tantalizing possibility of analytically seeking approximate solutions to specific problems and revealing various features of the series of solutions obtained.

Nonlinear differential equations are everywhere, and it is often useful for engineers to have an approximate closed-form solution for describing a special nonlinear problem. Various perturbation methods are widely used for this purpose, but results often deteriorate quickly as the degree of nonlinearity increases, which makes the finding of solutions rather demanding using new analytical methods. Consequently, if we are really determined to extract meanings from analytic formulations of physical processes, we must resort to amelioration of the classical perturbation methods using modern mathematical tools or even ancient mathematics with physical understanding such as homotopy, variational theory, iteration technology in numerical simulation, energy balance, and ancient Chinese mathematics, which we shall concentrate upon in this special issue.

In the last three decades, with the rapid development of nonlinear science, there has been ever-increasing interest from scientists and engineers in the analytical asymptotic techniques for addressing nonlinear problems. Though it is very easy for us now to find the solutions of linear systems by means of computers, it is still very difficult to solve nonlinear problems either numerically or analytically. This is possibly due to the fact that various discredited methods and numerical simulations apply iteration techniques to find numerical solutions of nonlinear problems, and nearly all iterative methods are sensitive to initial solutions [1], so it is very difficult to obtain converged results in cases of strong nonlinearity. In addition, most important information, such as the natural circular frequency of a nonlinear oscillation, depends on the initial condition (i.e. amplitude of oscillation), and will be lost during the procedure of numerical simulation. For example, numerical methods can easily solve the KdV6 equation given by

$$\left(\partial_x^3 + 8u_x\partial_x + 4u_{xx}\right)\left(u_t + u_{xxx} + 6u_x^2\right) = 0 \tag{1}$$

for any values of u, but the numerical solution cannot outline the dispersion relation among the wave speed, frequency and initial conditions, which plays a pivotal role in engineering. Though computer science is growing very fast, and numerical simulation is applied everywhere, non-numerical issues will still play a large role.

* Corresponding address: Donghua University, 1882 Yan'an Xilu Road, Shanghai 200051, China.
E-mail addresses: jhhe@dhu.edu.cn (J.-H. He), ericlee@cityu.edu.hk (E.W.M. Lee).

1 Senior Research Fellow at the City University of Hong Kong.

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.03.001

For Eq. (1), the exp-function method can be successfully applied, where the solution is assumed to have the form [2,3]

$$u(x,t)=\frac{\sum_{n=-c}^{d} a_n \exp(q_n x+p_n t)}{\sum_{m=-f}^{g} b_m \exp(\alpha_m x+\beta_m t)} \tag{2}$$

where $a_i, b_i\alpha_i, \beta_i q_i, p_i$ are unknown constants to be further determined. Please refer to [3] for the detailed solution procedure.

The key to physical understanding of real-life physics is the use of approximation. Most physical problems facing engineers, physicists, and applied mathematicians today exhibit certain essential features which preclude exact analytical solutions (except in the case of solitary solutions of nonlinear equations). Even if the exact solutions, with complicated and unfamiliar functions of the variable, can be found explicitly, they may be useless for mathematical and physical interpretation or numerical evaluation. By contrast, approximate solutions can strip away the overlying detail to show the essential relationships between the physical variables that are familiar to all scientists and engineers.

As an illustration, consider an equation that can be used as a mathematical model for a chemical reaction [4]:

$$A+2B\to C.$$

Assume that the numbers of molecules A and B are a and b respectively at $t=0$, and the number of molecules C at time t is c; the equation can be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}t}=K(a-u)(b-2u)^2,\quad u(0)=0. \tag{3}$$

The exact solution reads [4]

$$t=\frac{1}{K}\left\{\frac{1}{2a-b}\left(\frac{1}{b-2u}-\frac{1}{b}\right)+\frac{2}{(2a-b)^2}\ln\left(\frac{1-2u/b}{1-u/a}\right)\right\}. \tag{4}$$

This expression is too complicated to give a direct answer to most practical questions about the reaction, such as "What does the curve of u against t look like?" or "How does the shape of this curve depend upon a, b, and K, the constants in the equation?" [4]. By contrast, the approximate solution answers these questions directly. It reads

$$u\approx\frac{1}{2}b(1-\mathrm{e}^{-\beta t}), \tag{5}$$

where

$$\beta=Kab\left(1-\frac{1}{4a}\right). \tag{6}$$

In words, solution (5) states that "u increases approximately exponentially from zero at $t=0$ to a final value $u=b/2$, with a time constant given by formula (6)" [4].

Recently dramatic breakthroughs were made in analytical methods, e.g., the homotopy perturbation method and the variational iteration method. All the proposed methods are valid not only for weakly nonlinear equations, but also for strongly nonlinear ones, and the solutions obtained are valid for the whole solution domain.

This special issue also includes some papers on ancient Chinese mathematics; see the papers by Yue-Yun Shen, Hui-Li Zhang, Ji-Huan He, Jie Fan, and Ling Zhao. Great classics, when revisited in the light of new developments, may reveal hidden pearls, as is the case with ancient Chinese methods and He Chengtian's inequality.

The reader can easily comprehend the theories suggested in this special issue with a basic knowledge of advanced calculus. Each paper in this special issue contains beautiful worked examples showing how to obtain analytical solutions to previously intractable nonlinear differential equations that describe the behavior of real systems. Each paper delights in showing how easy it is to use the new methods across a wide area of mathematical physics. The given examples can be used as paradigms for many other applications in searching for analytical solutions for real-life problems arising from biology, economics, chemistry, physics, mechanics, and others.

It is very important to get full insight into the physical understanding of a special problem when we use the new methods. For Eq. (3), we can outline a qualitative sketch, and a trial function solution can be given in the form

$$u_0=\lambda(1-\mathrm{e}^{-\beta t}) \tag{7}$$

where β and λ are unknown constants to be further determined.

Eq. (7) is the solution of the following linearized equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\beta(\lambda-u),\quad u(0)=0. \tag{8}$$

If the homotopy perturbation method is applied, then we can construct a homotopy in the form

$$(1-p)\left\{\frac{\mathrm{d}u}{\mathrm{d}t}-\beta(\lambda-u)\right\}+p\left\{\frac{\mathrm{d}u}{\mathrm{d}t}-K(a-u)(b-2u)^2\right\}=0, \quad u(0)=0. \tag{9}$$

When $p = 0$, the solution of Eq. (9) is Eq. (7); when $p = 1$, it turns out to be the original one. Generally first-order approximation is enough, and various methods can be used to identify β; please refer to [5–7] for the detailed solution procedure.

If the variational iteration method is used, we can use the following iteration formulation:

$$\begin{cases} u^{(n)}+f(u,u',u'',\dots,u^{(n)})=0 \\ u_{n+1}(t)=u_0(t)+(-1)^n\displaystyle\int_0^t \frac{1}{(n-1)!}(s-t)^{n-1}f(u_n,u_n',u_n'',\dots,u_n^{(n)})\mathrm{d}s. \end{cases} \tag{10}$$

A suitable choice of initial guess is also very important [8–10].

If we resort to the parameter-expansion method [7], we can rewrite Eq. (3) in the form

$$\frac{\mathrm{d}u}{\mathrm{d}t}-0+0\cdot u-K(a-u)(b-2u)^2=0. \tag{11}$$

We expand the solution into the form

$$u=u_0+u_1p+u_2p^2+\cdots \tag{12}$$

where p is a bookkeeping parameter, $p = 1$.

The first zero, second zero, and K in the left hand side of Eq. (11) can be expanded respectively as follows:

$$0=\lambda\beta+a_1p+a_2p^2+\cdots \tag{13}$$

$$0=\beta+b_1p+b_2p^2+\cdots \tag{14}$$

and

$$K=K_1p+K_2p^2+\cdots. \tag{15}$$

Substituting Eqs. (12)–(15) into (11), and proceeding in the same way as in the classical perturbation method, we can obtain a series of linear equations that can easily be solved. Here we write down the zero-order equation:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t}-\beta(\lambda-u_0)=0, \quad u_0(0)=0 \tag{16}$$

which is exactly Eq. (8); this is the very reason for the parameters in (11) being expanded in different ways, as given in Eqs. (12)–(15). Refer to [11–14] for the detailed solution procedure.

Acknowledgments

This special issue "New analytical methods and applications" of *Computers and Mathematics with Applications* conveys a strong, reliable, efficient, and promising development of various new methods. Special thanks go to the referees and co-guest editors, Dr. Lan Xu, Prof. Augusto Beléndez, Prof. P. Donald Ariel, for their valuable work; without their help, the achievement of this special issue would have been impossible. We here thank Dr. Ervin Y. Rodin and Dr. Christee Zimmermann for providing us with the opportunity to produce this special issue on these exciting methods.

This work received support from City University of Hong Kong and Donghua University.

References

[1] J.G. Ning, C. Wang, T.B. Ma, Numerical analysis of the shaped charged jet with large cone angle, Int. J. Nonlinear Sci. Numer. 7 (2006) 71–78.
[2] X.W. Zhou, Y.X. Wen, J.H. He, Exp-function method to solve the nonlinear dispersive $K(m, n)$ equations, Int. J. Nonlinear Sci. Numer. 9 (2008) 301–306.
[3] J.H. He, An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering, Internat. J. Modern Phys. B 22 (21) (2008) 3487–3578.
[4] J.R. Acton, P.T. Squire, Solving Equations with Physical Understanding, Adam Hilger Ltd, Bristol, Boston, 1985.
[5] J.H. He, Recent development of the homotopy perturbation method, Topol. Methods Nonlinear Anal. 31 (2008) 205–209.
[6] J.H. He, New interpretation of homotopy perturbation method, Internat. J. Modern Phys. B 20 (2006) 2561–2568.
[7] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.
[8] J.H. He, X.H. Wu, Variational iteration method: New development and applications, Comput. Math. Appl. 54 (2007) 881–894.
[9] J.H. He, Variational iteration method — Some recent results and new interpretations, J. Comput. Appl. Math. 207 (2007) 3–17.
[10] R. Mokhtari, Variational iteration method for solving nonlinear differential–difference equations, Int. J. Nonlinear Sci. Numer. 9 (2008) 19–23.
[11] S.Q. Wang, J.H. He, Nonlinear oscillator with discontinuity by parameter-expansion method, Chaos Solitons Fractals 35 (2008) 688–691.
[12] D.H. Shou, J.H. He, Application of parameter-expanding method to strongly nonlinear oscillators, Int. J. Nonlinear Sci. Numer. 8 (2007) 121–124.
[13] L. Xu, He's parameter-expanding methods for strongly nonlinear oscillators, J. Comput. Appl. Math. 207 (2007) 148–154.
[14] F.O. Zengin, M.O. Kaya, S.A. Demirbag, Application of parameter-expansion method to nonlinear oscillators with discontinuities, Int. J. Nonlinear Sci. Numer. 9 (2008) 267–270.